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ON OPTIMUM FINITE ELEMENT GRIDS FOR A CHORD

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Abstract—The variational approach to grid optimization in the finite element method is applied on a simple chord and the grid geometry is considered as a primary parameter too. As a consequence, two sets of equations result, one the usual equilibrium and another, the physical meaning of which is not clear in general. Under consideration are eigenoscillations and static loading of a chord. In spite of both problems being linear in a continuum formulation, the FEM formulation becomes nonlinear by the variation of the grid coordinates. So problems of uniqueness arise. The optimal solutions are presented. But the danger is shown to run into local optimum solutions which are worse than those without grid optimization.

1. INTRODUCTION

As proposed by Carroll and Barker[1], Turcke and McNeice[2] and Prager[3] in finite element idealization a true minimum of system potential energy should consider the grid coordinates as primary parameters too. It was intended to apply this method on membranes under very small prestress (like yacht sails). Difficulties in this field were the reason why a simple chord was investigated.

In Ref. [1] we read:

> "The question of uniqueness still remains. Although indications are that a unique set of l_i's was found for each problem solved, there could be problems for which several configurations do exist such that the residue vector r_k will vanish for some nonminimum set of l_i's."

Do such misleading solutions exist for a chord?

2. OPTIMUM FINITE-ELEMENT FORMULATION FOR A CHORD

The potential energy Π and the kinetic energy E for a chord prestressed with a force H are

$$\Pi = \frac{H}{2}\int (w')^2 \, dx - \int qw \, dx \tag{1a}$$

$$E = \frac{\mu}{2}\int (\dot{w})^2 \, dx \tag{1b}$$

where $q(x)$ denotes a static loading.

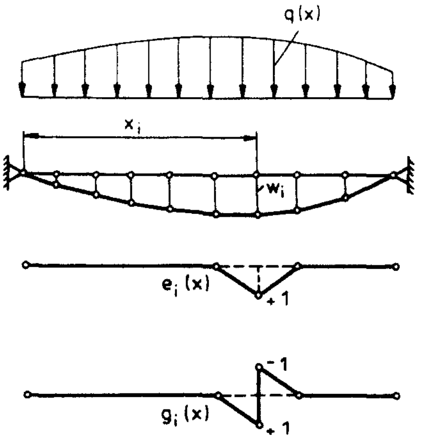

Fig. 1. Discretization of the chord.

Let us use n linear finite elements by approximating w by the estimate function $\hat{w}$ (see Fig. 1)

$$\hat{w}(x) = \sum_{i=1}^{n-1} w_i e_i(x) \tag{2a}$$

with the discrete parameters w_i

$$w_i = w(x = x_i) \quad \text{and} \quad w_0 = w_n = 0 \tag{2b}$$

where the discretization coordinates x_i

$$x_{i-1} < x_i < x_{i+1} \tag{2c}$$

are considered as primary parameters too.

With equation (2a) the energies, eqns (1a) and (1b), yield

$$\Pi = \frac{H}{2} \sum_{i=1}^{n} (x_i - x_{i-1})(w_i')^2 - \sum_{i=1}^{n-1} w_i \int q(x) e_i(x) \, \mathrm{d}x \tag{3a}$$

with

$$w_i' = \frac{w_i - w_{i-1}}{x_i - x_{i-1}} \tag{3b}$$

$$E = \frac{\mu}{2} \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{3} (\dot{w}_i^2 + \dot{w}_i \dot{w}_{i-1} + \dot{w}_{i-1}^2). \tag{3c}$$

Considering the static problem we find the set of equilibrium equations

$$\frac{\partial \Pi}{\partial w_i} = 2Hd_i - \int q(x) e_i(x) \, \mathrm{d}x = 0 \tag{4a}$$

where the abbreviation d_i denotes

$$d_i = \tfrac{1}{2}(w_i' - w_{i+1}') \tag{4b}$$

and a second set of equations

$$\frac{\partial \Pi}{\partial x_i} = -\frac{w_i'+w_{i+1}'}{2}\left(2Hd_i - \int q(x)e_i(x)\,\mathrm{d}x\right) + d_i \int q(x)g_i(x)\,\mathrm{d}x = 0. \tag{4c}$$

For the problem of harmonic oscillations $q(x)$ is regarded to be zero and instead of eqn (2a) we assume with the eigenfrequency ω

$$w(x,t) = \sin(\omega t) \sum_{i=1}^{n-1} w_i e_i(x) \tag{5a}$$

and receive with the sound velocity c

$$c^2 = \frac{H}{\mu} \tag{5b}$$

from Hamilton's principle

$$\int (E - \Pi)\,\mathrm{d}t \Rightarrow \min \tag{5c}$$

the two sets of equations

$$\omega^2\left[\frac{x_i - x_{i-1}}{3}(2w_i + w_{i-1}) + \frac{x_{i+1} - x_i}{3}(2w_i + w_{i+1})\right] - 4c^2 d_i = 0 \tag{6a}$$

$$\frac{\omega^2}{3}[w_i(w_{i-1} - w_{i+1}) + w_{i-1}^2 - w_{i+1}^2] + c^2(w_i'^2 - w_{i+1}'^2) = 0. \tag{6b}$$

Both sets, eqn (4) and (6), are badly conditioned, so that direct solution with a Newton–Raphson method will fail. Further suitable treatment of these equations is required.

3. DISCUSSION OF THE STATIC PROBLEM

Introducing the equilibrium equation, eqn (4a), into eqn (4c) and excluding d_i to be zero we now get the simple statement

$$\int q(x)g_i(x)\,\mathrm{d}x = 0. \tag{7}$$

Here the displacements w are eliminated and by the function $g_i(x)$ (see Fig. 1) eqn (7) is the equation for the grid coordinates x_i, which obviously depend on load distribution $q(x)$ only.

It is easy to show that for constant loading $q(x) = q_0$ the equidistant grid is a unique solution of eqn (7) under restriction (2c). For a loading proportional to x we get a unique

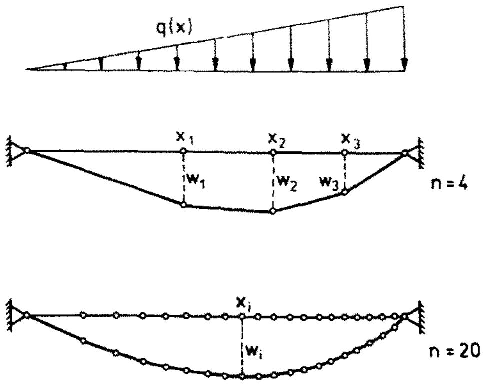

Fig. 2. Optimal grid and deflections for proportional loading.

non-equidistant grid, shown in Fig. 2 for 4 and 20 elements. The deflections w_i are computed without numerical problems from eqns (4a).

Considering loadings linear in x it is easy to show that eqns (7) and (2c) have non-unique solutions in all cases with a zero point of the loading within the range of the chord. But no examples are discussed here because they are very similar to those of the next chapter.

4. EIGENOSCILLATIONS

Equations (6a) and (6b) describe the eigenoscillation problem. Combining both and excluding again d_i to be zero we get the simple statement

$$(x_i - x_{i-1})(2w_i + w_{i-1}) - (x_{i+1} - x_i)(2w_i + w_{i+1}) = 0 \tag{8}$$

which is an equation of grid optimization.

Another derivation of eqn (8) is, to introduce the negative mass accelerations as static loading $q(x)$ into eqn (7) and to use Simpson's formula for integration.

Using eqns (6a) and (8) there are no numerical problems in finding the grid coordinates, the eigenfrequency and the eigenmodes. For the first eigenmode we find a unique optimal grid. Table 1 shows the affiliated eigenfrequency (referred to the exact value) and compares it with the equidistant-grid solution ω_1^e

Table 1. First eigenfrequency of a chord approximations related to the exact value $\omega_1 = \pi(c/l)$

Number of elements	Equidistant	Optimal grid
3	1.0461	1.0396
4	1.0259	1.0211
5	1.0165	1.0131
6	1.0115	1.0090
7	1.0084	1.0065
8	1.0064	1.0049
9	1.0051	1.0039
10	1.0041	1.0031
11	1.0034	1.0026
12	1.0029	1.0022
⋮	⋮	⋮
20	1.0010	1.0008

Table 2. Second eigenfrequency of a chord approximation related to the exact value of $\omega_2 = 2\pi(c/l)$

Number of elements	Equidistant	Optimal grid				
3	1.117	1.103		local optimum		
4	1.103	1.073				
5	1.066	1.040	1.064			
6	1.046	1.031	1.061			
7	1.034	1.021	1.027	1.060		
8	1.026	1.017	1.025	1.059		
9	1.020	1.013	1.015	1.024	1.058	
10	1.017	1.011	1.014	1.023	1.058	
11	1.014	1.009	1.010	1.013	1.023	1.057
12	1.011	1.008	1.009	1.013	1.022	1.057
⋮	⋮			⋮		
20	1.0041	1.0028...				

$$(w_\kappa^e)^2 = \left(\frac{c}{l}\right)^2 6n^2 \frac{1-\cos\dfrac{\kappa\pi}{n}}{2+\cos\dfrac{\kappa\pi}{n}} \qquad \kappa = 1,2,\ldots,n-1. \tag{9}$$

Of course the error of the optimal solution is smaller than that of the equidistant one with the same number of elements. But comparing solutions with the same number of unknowns it turns out that grid optimization does not pay. For example the six element equidistant solution is much better than the three element optimal solution. Even more interesting are the results for the second eigenmode shown in Table 2. Beginning with five elements we now find several "optimal" grids and according to this fact different approximations of the eigenfrequency. Some of these values being local optimums only, are worse than the approximations of the equidistant grid.

The reason for this odd behaviour is (similar to the static problem) the fact that there is a deflection zero point, the oscillation node, within the range of the chord. From eqn (8) and restriction (2c) it follows that an oscillation node cannot coincide with a grid point. Hence for an even number of elements the antisymmetric shape of the second eigenmode is optimally approximated by an unsymmetrical grid (see Fig. 3).

To get more information about the nonuniqueness of the grid we look at the affiliated eigenmodes for an eight element chord shown in Fig. 3. It is possible to push 4 or 5 or 6 grid points into the left part of the mode, the first case being the global optimum, and the other two being only local optima. Altogether there are 36 different optimal solutions for a chord with eight elements. They are shown in Table 3 affiliated to the corresponding

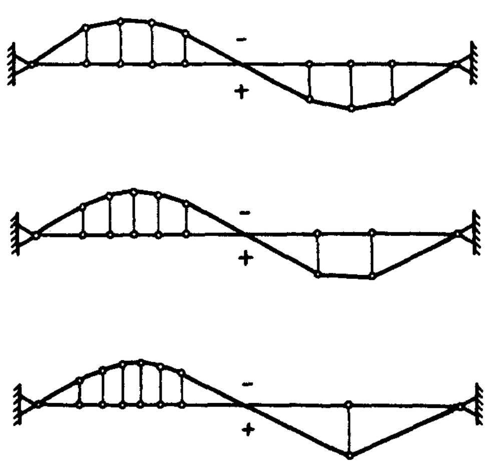

Fig. 3. Optimal eight element grids for the second eigenmode.

Table 3. Related eigenfrequencies for an eight element chord

Mode	Exact value	Equidistant	Optimal solutions									
1	π	3.162	3.157									
2	2π	6.45	6.39	6.44	6.65							
3	3π	9.97	9.741	9.742	9.90	9.91	9.92	9.937	9.944	10.12	10.14	
4	4π	13.86	13.28	13.41	13.420	13.427	13.428	13.438	13.446	13.59	13.62	
5	5π	18.12	16.93	16.9410	16.94190	16.94195	16.952	16.953	16.96	17.08	17.0983	17.0984
6	6π	22.52	20.595	20.6050	20.6051							
7	7π	26.20	24.25									

mode. The figures for the eigenfrequencies are given with as many digits as necessary to show the difference between the values.

5. CONCLUSION

The given examples of a chord show very clearly that grid optimization by variation of the grid coordinates is a problematic method.

The optimal grid equations, eqns (7) and (8), have a weak physical interpretation only.

The grid coordinates have to be chosen in such a manner, that the loading for each grid point by the shear forces is the same from both the neighbouring elements.

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